I) Vypočtěte integrály:

Substituce:

1) $$\int \underbrace{\frac{tg\,x}{\cos^2 x}}_{tg\,x \cdot \frac{1}{\cos^2 x}} dx = \left| \begin{matrix} t = tg\,x \\ dt = \frac{1}{\cos^2 x}\,dx \end{matrix} \right| = \int t\,dt = \frac{1}{2}t^2 + c = \underline{\underline{\frac{1}{2}tg^2 x + c}}$$

Per partes:

2) $$\int x \ln x\,dx = \left| \begin{matrix} u = \ln x & v' = x \\ u' = \frac{1}{x} & v = \frac{1}{2}x^2 \end{matrix} \right| = \frac{1}{2}x^2 \ln x - \int \frac{1}{x} \cdot \frac{1}{2}x^2\,dx =$$

$$= \frac{1}{2}x^2 \ln x - \frac{1}{2}\int x\,dx = \frac{1}{2}x^2 \ln x - \frac{1}{2}\cdot\frac{1}{2}x^2 + c = \underline{\underline{\frac{1}{2}x^2\left(\ln x - \frac{1}{2}\right) + c}}$$

3) $$\int \frac{x}{\underbrace{x^2+4x+4}} dx = *$$

Rozložíme na parciální zlomky:

$$\frac{x}{x^2+4x+4} = \underbrace{\frac{x}{(x+2)^2} = \frac{A}{x+2} + \frac{B}{(x+2)^2}}\quad / \cdot (x+2)^2$$

Rovnici vynásobíme společným jmenovatelem všech zlomků

$$x = A(x+2) + B = Ax + 2A + B$$

Porovnáme koeficienty u stejných mocnin x

$$\begin{cases} x^1: & 1 = A \\ x^0: & 0 = 2A + B \quad \Rightarrow \quad B = -2A = -2 \end{cases}$$

$$\frac{x}{x^2+4x+4} = \frac{1}{x+2} - \frac{2}{(x+2)^2}$$

$$* = \int \left(\underbrace{\frac{1}{x+2}}_{A} - \underbrace{\frac{2}{(x+2)^2}}_{B} \right) dx = \underline{\underline{\ln|x+2| + \frac{2}{x+2} + c}}$$

$$A: \int \frac{1}{x+2}\,dx = \left| \begin{matrix} t = x+2 \\ dt = dx \end{matrix} \right| = \int \frac{1}{t}\,dt = \ln|t| = \ln|x+2|$$

$$B: -\int \frac{2}{(x+2)^2}\,dx = \left| \begin{matrix} t = x+2 \\ dt = dx \end{matrix} \right| = -2\int \frac{1}{t^2}\,dt = -2\int t^{-2}\,dt = -2\,\frac{t^{-1}}{-1} = \frac{2}{t} = \frac{2}{x+2}$$

Per partes:

4) $$\int \underbrace{\operatorname{arctg} x}_{1 \cdot \operatorname{arctg} x}\,dx = \left| \begin{matrix} u = \operatorname{arctg} x & v' = 1 \\ u' = \frac{1}{1+x^2} & v = x \end{matrix} \right| = x \operatorname{arctg} x - \frac{1}{2}\int \frac{2x}{1+x^2}\,dx =$$

vzorec: $\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c$

$$= \underline{\underline{x \operatorname{arctg} x - \frac{1}{2}\ln(1+x^2) + c}}$$

substituce:

5) $$\int x e^{x^2}\,dx = \left| \begin{matrix} t = x^2 \\ dt = 2x\,dx \\ \frac{1}{2}dt = x\,dx \end{matrix} \right| = \frac{1}{2}\int e^t\,dt = \frac{1}{2}e^t + c = \underline{\underline{\frac{1}{2}e^{x^2} + c}}$$

6) $\int \sin x \cos^3 x \, dx = \left| \begin{array}{l} \text{Substituce:} \\ t = \cos x \\ dt = -\sin x \, dx \end{array} \right| = -\int t^3 dt = -\frac{1}{4} t^4 + c = \underline{\underline{-\frac{1}{4}\cos^4 x + c}}$

7) $\int \frac{1}{\underbrace{x^3+x}} dx = *$

Rozložíme na parciální zlomky:

$$\frac{1}{x^3+x} = \underbrace{\frac{1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}} \qquad / \cdot x(x^2+1)$$

$$1 = A(x^2+1) + (Bx+C)x = Ax^2 + A + Bx^2 + Cx$$

- x^2: $0 = A + B \Rightarrow B = -A = -1$
- x^1: $0 = C$
- x^0: $1 = A$

$$\frac{1}{x^3+x} = \frac{1}{x} - \frac{x}{x^2+1}$$

$$* = \int \left(\frac{1}{x} - \frac{\frac{1}{2} \cdot 2 \cdot x}{x^2+1} \right) dx = \underline{\underline{\ln|x| - \frac{1}{2}\ln(x^2+1) + c}}$$

8) $\int x \sin 3x \, dx = \left| \begin{array}{ll} u = x & v' = \sin 3x \\ u' = 1 & v = -\frac{1}{3}\cos 3x \end{array} \right| = -\frac{1}{3} x \cos 3x + \frac{1}{3} \overbrace{\int \cos 3x \, dx}^{\text{obdobně jako } v} =$

$$\left(v = \int \sin 3x \, dx = \left| \begin{array}{l} t = 3x \\ dt = 3dx \Rightarrow dx = \frac{1}{3} dt \end{array} \right| = \frac{1}{3} \int \sin t \, dt = -\frac{1}{3} \cos t = -\frac{1}{3} \cos 3x \right)$$

$$= -\frac{1}{3} x \cos 3x + \frac{1}{3} \cdot \frac{1}{3} \sin 3x + c = \underline{\underline{-\frac{1}{3} x \cos 3x + \frac{1}{9} \sin 3x + c}}$$

II) Vypočtěte integrály:

1) $\displaystyle\int_1^{e^2} \underbrace{\ln x}_{1\cdot\ln x}\,dx = \left| \begin{matrix} u=\ln x & v'=1 \\ u'=\frac{1}{x} & v=x \end{matrix} \right| = [x\cdot\ln x]_1^{e^2} - \int_1^{e^2} \frac{1}{x}\cdot x\,dx = [x\cdot\ln x]_1^{e^2} - \int_1^{e^2} dx =$

$= [x\cdot\ln x - x]_1^{e^2} = e^2\cdot\underbrace{\ln e^2}_{2} - e^2 - 1\cdot\underbrace{\ln 1}_{0} + 1 = \underline{\underline{e^2+1}}$

2) $\displaystyle\int_1^{e} \underbrace{\frac{1+\ln x}{x}}_{(1+\ln x)\cdot\frac{1}{x}}\,dx = \left| \begin{matrix} t=\ln x \\ dt=\frac{1}{x}dx \end{matrix} \quad \begin{array}{c|c|c} x & 1 & e \\ \hline t & 0 & 1 \end{array} \right| = \int_0^1 (1+t)\,dt = \left[t+\frac{t^2}{2}\right]_0^1 =$

↳ při použití substituce musíme transformovat meze:
$x=1 \to t=\ln 1 = 0$
$x=e \to t=\ln e = 1$

$= 1+\frac{1}{2} = \underline{\underline{\frac{3}{2}}}$

3) $\displaystyle\int_0^1 x\,\mathrm{arctg}\,x\,dx = \left| \begin{matrix} u=\mathrm{arctg}\,x & v'=x \\ u'=\frac{1}{1+x^2} & v=\frac{x^2}{2} \end{matrix} \right| = \left[\frac{1}{2}x^2\mathrm{arctg}\,x\right]_0^1 - \frac{1}{2}\int_0^1 \underbrace{\frac{x^2+1-1}{1+x^2}}_{\frac{x^2+1}{1+x^2}-\frac{1}{1+x^2}=1-\frac{1}{1+x^2}}\,dx =$

$= \left[\frac{1}{2}x^2\mathrm{arctg}\,x\right]_0^1 - \frac{1}{2}\int_0^1 \left(1-\frac{1}{1+x^2}\right)dx =$

$= \left[\frac{1}{2}x^2\mathrm{arctg}\,x\right]_0^1 - \frac{1}{2}\left[x-\mathrm{arctg}\,x\right]_0^1 = \frac{1}{2}\left[x^2\mathrm{arctg}\,x - x + \mathrm{arctg}\,x\right]_0^1 =$

$= \frac{1}{2}\left(1\cdot\underbrace{\mathrm{arctg}\,1}_{\frac{\pi}{4}} - 1 + \underbrace{\mathrm{arctg}\,1}_{\frac{\pi}{4}} - 0\cdot\underbrace{\mathrm{arctg}\,0}_{0} + 0 - \underbrace{\mathrm{arctg}\,0}_{0}\right) =$

$= \frac{1}{2}\left(\frac{\pi}{4}-1+\frac{\pi}{4}\right) = \frac{1}{2}\left(\frac{\pi}{2}-1\right) = \underline{\underline{\frac{1}{4}(\pi-2)}}$

4) $\displaystyle\int_1^3 \underbrace{\frac{1}{x^2+3x+2}}_{\text{Rozložíme na parciální zlomky}}\,dx = *$

$$\frac{1}{x^2+3x+2} = \frac{1}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}$$

$$1 = A(x+2)+B(x+1) = Ax+2A+Bx+B$$

x^1: $0 = A+B$ $\quad /\cdot(-1)$ $\quad \Rightarrow \quad B=-A=-1$
x^0: $1 = 2A+B$
$1 = A$

$$\frac{1}{x^2+3x+2} = \frac{1}{x+1} - \frac{1}{x+2}$$

$* = \displaystyle\int_1^3 \left(\frac{1}{x+1}-\frac{1}{x+2}\right)dx = \left[\ln|x+1| - \ln|x+2|\right]_1^3 = \left[\ln\left|\frac{x+1}{x+2}\right|\right]_1^3 =$

$= \ln\frac{3+1}{3+2} - \ln\frac{1+1}{1+2} = \ln\frac{4}{5} - \ln\frac{2}{3} = \ln\frac{\frac{\cancel{4}^{\,2}}{5}}{\frac{\cancel{2}^{\,1}}{3}} = \underline{\underline{\ln\frac{6}{5}}}$

5) $\int_0^1 x e^x dx = \left| \begin{matrix} u = x & v' = e^x \\ u' = 1 & v = e^x \end{matrix} \right| = [x e^x]_0^1 - \int_0^1 e^x dx = [x e^x - e^x]_0^1 =$

$= 1 \cdot e^1 - e^1 - 0 \cdot e^0 + e^0 = e - e + 1 = \underline{\underline{1}}$

6) $\int_0^1 x \sqrt{1-x^2}\, dx = \left| \begin{matrix} t = 1 - x^2 \\ dt = -2x\, dx \\ -\frac{1}{2} dt = x\, dx \end{matrix} \qquad \begin{array}{c|c|c} x & 0 & 1 \\ \hline t & 1 & 0 \end{array} \right| = -\frac{1}{2} \int_1^0 \sqrt{t}\, dt =$

$= \frac{1}{2} \int_0^1 t^{\frac{1}{2}} dt = \frac{1}{2} \left[\frac{t^{\frac{3}{2}}}{\frac{3}{2}} \right]_0^1 = \frac{1}{3} [t \sqrt{t}]_0^1 = \frac{1}{3}(1 - 0) = \underline{\underline{\frac{1}{3}}}$

$t^{\frac{3}{2}} = \sqrt{t^3} = t\sqrt{t}$

III) Vypočtěte parciální derivace prvního řádu a výsledky upravte.

1) $f(x,y) = x\cdot\cos y - y^2\cdot\sin x + \ln x - e^{2y}$

$f'_x = 1\cdot\cos y - y^2\cdot\cos x + \frac{1}{x} - 0 = \underline{\underline{\cos y - y^2\cos x + \frac{1}{x}}}$

$f'_y = x\cdot(-\sin y) - 2y\cdot\sin x + 0 - e^{2y}\cdot 2 = \underline{\underline{-x\sin y - 2y\sin x - 2e^{2y}}}$

2) $f(x,y) = x\cdot e^{\frac{y}{x}}$

$f'_x = 1\cdot e^{\frac{y}{x}} + x\cdot \underbrace{e^{\frac{y}{x}}\cdot y\cdot(-x^{-2})} = e^{\frac{y}{x}} - \frac{xy}{x^2}e^{\frac{y}{x}} = \underline{\underline{e^{\frac{y}{x}}\left(1-\frac{y}{x}\right)}}$

– derivace součinu

derivace složené funkce

$\frac{y}{x} = y\cdot x^{-1}$

$f'_y = x\cdot e^{\frac{y}{x}}\cdot\frac{1}{x} = \underline{\underline{e^{\frac{y}{x}}}}$

3) $f(x,y) = \operatorname{arctg}\frac{x-y}{x+y}$

$$f'_x = \frac{1}{1+\left(\frac{x-y}{x+y}\right)^2}\cdot\frac{1\cdot(x+y)-(x-y)\cdot 1}{(x+y)^2} = \frac{1}{\frac{(x+y)^2+(x-y)^2}{(x+y)^2}}\cdot\frac{x+y-x+y}{(x+y)^2} =$$

$$= \frac{2y}{x^2+2xy+y^2+x^2-2xy+y^2} = \frac{2y}{2x^2+2y^2} = \underline{\underline{\frac{y}{x^2+y^2}}}$$

$$f'_y = \frac{1}{1+\left(\frac{x-y}{x+y}\right)^2}\cdot\frac{-1\cdot(x+y)-(x-y)\cdot 1}{(x+y)^2} = \frac{1}{\frac{(x+y)^2+(x-y)^2}{(x+y)^2}}\cdot\frac{-x-y-x+y}{(x+y)^2} =$$

$$= \frac{-2x}{x^2+2xy+y^2+x^2-2xy+y^2} = \frac{-2x}{2x^2+2y^2} = \underline{\underline{-\frac{x}{x^2+y^2}}}$$

4) $f(x,y) = \ln\frac{x^2+y^2}{x^2-y^2}$

$$f'_x = \frac{1}{\frac{x^2+y^2}{x^2-y^2}}\cdot\frac{2x(x^2-y^2)-(x^2+y^2)\cdot 2x}{(x^2-y^2)^2} = \frac{2x(x^2-y^2-x^2-y^2)}{(x^2+y^2)(x^2-y^2)} = \frac{-4xy^2}{x^4-y^4} = \underline{\underline{\frac{4xy^2}{y^4-x^4}}}$$

$$f'_y = \frac{1}{\frac{x^2+y^2}{x^2-y^2}}\cdot\frac{2y(x^2-y^2)-(x^2+y^2)\cdot(-2y)}{(x^2-y^2)^2} = \frac{2y(x^2-y^2+x^2+y^2)}{(x^2+y^2)(x^2-y^2)} = \underline{\underline{\frac{4x^2y}{x^4-y^4}}}$$